EXTREME POSITIVE OPERATORS ON $C(X)$ WHICH COMMUTE WITH GIVEN OPERATORS

BY

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0. **Introduction.** Let X be a compact Hausdorff space and $C(X)$ the space of continuous real functions on X. Let D_1 be the convex set of all linear operators T on $C(X)$ such that $T1=1$ and $T \geqq 0$, i.e. $Tf \geqq 0$ whenever $f \geqq 0$. It was first shown by A. and C. Ionescu Tulcea [8] and [11] that the extreme points of D_1 are precisely the multiplicative elements of D_1, that is, those T in D_1 satisfying

$$T(fg) = TfTg, \qquad f, g \text{ in } C(X).$$

Related results and extensions have been obtained for various sets of operators on various classes of algebras of functions (see [1] and its references). In the present paper, the author considers a somewhat different extension of this result. The author wishes to express his gratitude to Professors A. and C. Ionescu Tulcea for having indirectly suggested the problem and to Professor R. R. Phelps for his direction of the thesis from which this paper is drawn.

Let S be any semigroup of operators in D_1. For f in $C(X)$ and s in S, we let f_s be the image of f under s. We define D to be the convex set of all operators in D_1 which commute with the elements of S:

$$D = \{T \text{ in } D_1 : T(f_s) = (Tf)_s, f \text{ in } C(X) \text{ and } s \text{ in } S\}.$$

A natural question is whether the extreme points of D are multiplicative. A. Ionescu Tulcea has shown [7, Remark (2), p. 824] that this is indeed the case if X is itself a topological group and S is given by left-translations by members of X, i.e. $f_s(x)=f(sx)$, s and x in X. Our classification of extreme points of D will show that the above question has a negative answer in our more general context even if S is a finite group. However, an affirmative answer can be given for semigroups which satisfy certain compactness conditions, provided the problem is restated as follows.

For any operator T on $C(X)$, let T_x be the functional on $C(X)$ defined by $T_x f = Tf(x)$, $f \in C(X)$. Let S^x be the subset of S which fixes x in the following sense:

$$S^x = \{s \text{ in } S : f_s(x) = f(x), f \text{ in } C(X)\}.$$

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For any subset S_1 of S, let $P(S_1)$ be the set of S_1-invariant probability measures on S:

$$P(S_1) = \{\mu \text{ in } C^*(X) : \mu 1 = 1, \mu \geqq 0, \text{ and } \mu(f_s) = \mu f, f \text{ in } C(X) \text{ and } s \text{ in } S_1\}.$$

The result of A. Ionescu Tulcea mentioned above is (for special S and X)

(C) If T is extreme in D, then T_x is extreme in $P(S^x)$ for all x in X.

Notice that the converse follows easily from the fact that T_x is in $P(S^x)$ for all x in X whenever T is in D.

Our central result, Theorem 1, asserts that (C) is true provided S is a group of invertible operators which is compact in the weak operator topology. Theorems 2 and 3 give different extensions of the result of A. Ionescu Tulcea. We show by examples that some compactness for S is required; in Example 2, for instance, conclusion (C) is shown to fail rather drastically if S is taken to be the group of integers acting in an appropriate way on $C(X)$, where X is the cylinder $[0, 1] \times R \pmod{2\pi}$.

1. **Homeomorphisms and operators.**

Most of our results will concern invertible operators, partly because of the following lemma.

LEMMA 1. *If S is a group of invertible operators in D_1, then there exists a group H of homeomorphisms of X and an isomorphism $s \to h_s$ of S onto H such that $f_s = f \circ h_s$ for all f in $C(X)$.*

Proof. For s in S, let s^* be the adjoint map from $C^*(X)$ into $C^*(X)$; obviously s^* defines an affine map from the set P_1 of probability measures on X into P_1. But s is invertible, so s^* is invertible. Thus s^* is 1:1, maps P_1 onto P_1 and carries extreme points of P_1 onto extreme points of P_1. Because the extreme points of P_1 are point masses, the conclusion follows.

We will not distinguish between a group of invertible operators in D_1 and the corresponding group of homeomorphisms. Notice that if S is a group of homeomorphisms, then

$$S^x = \{s \text{ in } S : sx = x\}.$$

If S is a group of homeomorphisms, it is natural to consider at least three topologies on S: the strong operator topology, the weak operator topology, and the topology of pointwise convergence on X. We show below that these topologies agree if S is compact in any (and hence every) one of the three. This is a corollary to the following result due to R. Ellis [6]. (We wish to thank Professor H. Corson for calling this result to our attention.)

THEOREM A (ELLIS). *Suppose X is a locally compact, Hausdorff space and S is a group of homeomorphisms of X. Let S be given a topology in which the following maps are separately continuous:*

π: $X \times S \to X$ *given by* $\pi(x, s) = sx$,

φ: $S \times S \to S$ *given by* $\varphi(s, t) = st$.

Then π is jointly continuous. In particular, if X is a group and S is left-translation by members of X, then $S=X$ is a topological group with this topology.

COROLLARY. *Suppose S is a group of homeomorphisms which is compact in any of the following topologies: the strong operator topology, the weak operator topology, and the topology of pointwise convergence on X. Then these topologies agree on S.*

Proof. The equivalence of the pointwise topology and the weak operator topology follows from Corollary 8.12.8 (to Eberlein's Theorem) in [5]. The equivalence of the weak operator and strong operator topologies follows from Corollary 8.2 in [4].

2. **Definitions and notation.** Let S be a group of homeomorphisms of X. For x in X and f in $C(X)$, let $S(x)$ and $S(f)$ be the S orbits of x and f respectively, i.e.

$$S(x) = \{sx : s \in S\}, \qquad S(f) = \{f_s : s \in S\}.$$

The group S and any subset Y of X determine a subalgebra of $C(X)$,

$$A(S, Y) = \{f \text{ in } C(X) : f(sx) = f(x), s \text{ in } S, x \text{ in } Y\}.$$

Clearly $A(S, Y)$ is a closed subspace of $C(X)$. Because S is given by functions from X into X, $A(S, Y)$ is an algebra. If we define the equivalence relation $x \sim y$ if and only if $f(x) = f(y)$ for all f in $A(S, Y)$, then $A(S, Y)$ is (isometrically isomorphic to) $C(X/\sim)$ where $X/\sim$ is the set of equivalence classes of X with the quotient topology. Since $A(S, Y)$ separates points of $X/\sim$, it is a compact Hausdorff space. If $Y=X$ and S is compact on X (in any one of the topologies mentioned in Lemma 1), then it is straightforward to check that $y \sim x$ if and only if $y \in S(x)$ so that $X/\sim$ is the orbit space for S. A subset Y of X is said to be *invariant under S* if $S(x) \subset Y$ whenever $x \in Y$. For μ in $C^*(X)$, let supp μ denote the closed support of μ. It is easily seen that supp μ is invariant under S whenever μ is in $P(S)$. If Y is invariant under S, we may consider S as a group of homeomorphisms of Y. The topology of pointwise convergence on Y makes sense even though S may not be Hausdorff in this topology. We may factor out the (necessarily normal on Y) subgroup of all those s in S which equal the identity on Y. The resulting group with the quotient topology is again a group of homeomorphisms of Y and the quotient topology is the topology of pointwise convergence on Y. We will call this group with this topology *S restricted to Y*. If S restricted to Y is compact, we say S is *compact on Y*. In particular, S is compact on X if and only if S is compact in the topology of pointwise convergence on X. Theorem A implies that if S is compact on Y, then S restricted to Y is a topological group.

3. **Extreme points of $P(S)$.** For most of the following results, we need the classification of extreme points of $P(S)$ which is implied by Proposition 1 if S is compact on X.

Proposition 1. *Suppose S is a group of invertible operators in D_1 and that μ is an element of $P(S)$. Let Y be any closed subset of X which is invariant under S and contains* supp μ. *Let S be compact on Y and ds be normalized Haar measures for S restricted to Y. Then the following are equivalent:*

(i) *μ is extreme in $P(S)$.*

(ii) *$\mu f \mu g = \mu(fg)$ whenever f or g is in $A(S, Y)$, f and g in $C(X)$.*

(iii) *$\mu(g^2) = (\mu g)^2$ whenever g is in $A(S, Y)$ and $0 \leqq g \leqq 1$.*

(iv) *For each y in* supp μ, *$S(y) =$* supp μ.

(v) *For each f in $C(X)$ and y in* supp μ, *$\mu f = \int f(sy)\, ds$.*

Proof. (i) implies (ii). This is fairly standard; we include the proof for completeness. Suppose (ii) is false so that $\mu f \mu g - \mu(fg) \neq 0$ for some f in $C(X)$ and g in $A(S, Y)$. We may assume without loss of generality that $0 \leqq g \leqq 1$, since μ is linear and $\mu 1 = 1$. Define ν in $C^*(X)$ by

$$\nu h = \mu h \mu g - \mu(hg), \quad h \text{ in } C(X).$$

Since $0 \leqq g \leqq 1$, $\mu \pm \nu \geqq 0$. Clearly $(\mu \pm \nu)1 = 1$. Since g is in $A(S, Y)$ and supp μ is contained in Y, $\nu(h_s) = \nu h$ for all s in S and h in $C(X)$. Here we use the fact that S is given by homeomorphisms so that $(hg)_s = h_s g_s$. Therefore $\mu \pm \nu$ are in $P(S)$. But $\nu f \neq 0$, so μ is not extreme in $P(S)$.

(ii) implies (iii). This is trivial.

(iii) implies (iv). Suppose there exists y in supp μ such that $S(y) \neq$ supp μ. Let z be any point in supp μ, but not in $S(y)$. Let K_1 be a compact neighborhood of z in Y disjoint from $S(y)$ and let $K = \{sx : s \text{ is in } S, \text{ and } x \text{ is in } K_1\}$. Then K is a neighborhood of $S(z)$, disjoint from $S(y)$, which is compact by Theorem A. Hence there exists a k in $C(Y)$, $0 \leqq k \leqq 1$, such that k is 1 on K and 0 on $S(y)$. Let H be the closed convex hull, in $C(Y)$, of $S(k)$. Then H is compact since $S(k)$ is compact by the corollary to Theorem A. As a group of operators, S is then a compact group of affine maps of H into itself. Hence, by a theorem of Day [3], there is a fixed point, i.e. there is an h' in H such that $h'_s = h'$ for all s in S. Let h in $C(X)$ be any extension of h' to X such that $0 \leqq h \leqq 1$. Then, for all positive integers n, h^n is in $A(S, Y)$, $\mu(h^n) \geqq \mu(K) > 0$, and $\mu(h) < 1$. Hence there exists a positive integer n such that $\mu((h^n)^2) \neq (\mu(h^n))^2$. Let g be h^n for any such n.

(iv) implies (v). Suppose $S(y) =$ supp μ and f is in $C(X)$. Define h on (supp μ) $\times$ (S restricted to Y) by $h(x, s) = f(sx)$. By Theorem A, h is continuous on the product space and hence Fubini's theorem applies. Therefore,

$$(*) \qquad \int_{S(y)} \left[\int_S h(x, s)\, ds \right] d\mu(x) = \int_S \left[\int_{S(y)} h(x, s)\, d\mu(x) \right] ds.$$

But the left side of (*) is equal to $\int_S f(sy)\, ds$, since

$$\int_S h(x, s)\, ds = \int_S f(sx)\, ds = \int_S f(ss'y)\, ds = \int_S f(sy)\, ds,$$

and the right side of (*) is equal to μf since

$$\int_{S(y)} h(x, s)\, d\mu(x) = \int_{S(y)} f(sx)\, d\mu(x) = \mu(f_s) = \mu f.$$

(v) implies (iv). This is clear.

(iv) implies (i). Suppose that $\mu \pm \nu$ are in $P(S)$ and that $\operatorname{supp} \mu = S(y)$. Since $\mu \pm \nu \geqq 0$, $\operatorname{supp} \nu$ is contained in $\operatorname{supp} \mu$ and thus $\operatorname{supp} (\mu+\nu) = \operatorname{supp} (\mu-\nu) = S(y)$. Therefore, by "(iv) implies (v)," $(\mu+\nu)f = (\mu-\nu)f$ for all f in $C(X)$. Thus $\nu = 0$ and μ is extreme.

REMARK. The equivalence of (i) and (v) above is not too surprising if we consider the following fact. Let $Y = X$, P_1 be the set of all probability measure on X, and let ds represent Haar measure on S. Then $P(S) = \{\mu * ds : \mu \in P_1\}$ where $(\mu * ds)(f) = \int \mu(f_s)\, ds$, f in $C(X)$. Thus the equivalence of (i) and (v) just says that μ is extreme in $P(S)$ if and only if μ is the image of an extreme point in P_1, i.e. $\mu = \varepsilon_x * ds$ for some x in X.

In the case where S is a left-amenable semigroup of operators, the extreme points of $P(S)$ have been characterized by S. P. Lloyd [9] and a particularly simple proof has been given in [2]. These characterizations do not immediately yield the more detailed description given above when S is a group, however.

4. **Extreme operators in D.**

Theorem 1 below is trivial if we add the hypothesis that S is transitive on X, i.e. given x, y in X there is an s in S such that $sx = y$. This is because the condition that $T(f_s) = (Tf)_s$ just becomes the condition that $s * T_x = T_{sx}$. For more details, see Theorem 3. The real problem in Theorem 1 is generalizing this process when S is not transitive on X.

THEOREM 1. *If S is a group of homeomorphisms which is compact on X, then* (C) *is true.*

Proof. Suppose T is in D and T_y is not extreme in $P(S^y)$ for some point y in X. By Proposition 1, there is a g in $A(S^y, X)$ such that $T_y(g^2) \neq (T_y g)^2$ and $0 \leqq g \leqq 1$. Consider the set-valued map η from X to $C(X)$ defined by $\eta(x) = \{g_s\}$ if $sx = y$ for some s in S, and $\eta(x) =$ uniform closed convex hull of $S(g)$ otherwise. The map η is well defined since $sx = y = tx$ *implies* that $ts^{-1}y = y$. Since g is in $A(S^y, X)$, we have $g_s = g_{(ts^{-1})s} = g_t$.

We wish to use one of E. Michael's selection theorems [10]. Since X is paracompact and $C(X)$ is a Banach space, we need only show that η is lower semicontinuous (l.s.c.), i.e. if x_α tends to x and g' is in $\eta(x)$, then there exist g_α in $\eta(x_\alpha)$ such that g_α tends to g'. Since $S(y)$ is closed in X and η is clearly l.s.c. on the complement, it suffices to show that η is l.s.c. at each point x in $S(y)$ and $g' \in \eta(x)$ so that $g' = g_s$ where $sx = y$. If x_α tends to x, then for those x_α not in $S(y)$ we can let $g_\alpha = g'$. Thus we need only show that if $s_\alpha^{-1}y$ tends to $s^{-1}y$, then g_{s_α} tends to g_s. By taking subnets if necessary, we may suppose s_α tends to s pointwise, since S is compact on X. By the corollary to Theorem A, g_{s_α} tends to g_s. Hence it follows from [10] that there is a continuous map φ from X into $C(X)$ such that $\varphi(x)$ is in $\eta(x)$ for all x in X.

For each x in X, define g^x by

$$g^x(z) = \int \varphi(sx)(sz)\, ds \qquad (z \text{ in } X)$$

where ds is normalized Haar measure on S. Since $\varphi(X) \subset \{f : 0 \leqq f \leqq 1\}$, we have $0 \leqq g^x \leqq 1$. Because $\varphi(s^{-1}y) = g_s$, we have $g^y = g$; furthermore, $g_s^{sx} = g$ for all x in X and s in S because

$$\int \varphi(s's)(s's)\, ds' = \int \varphi(s'x)(s'z)\, ds', \quad \text{for all } z \text{ in } X.$$

We will show below that $x \to g^x$ is a continuous map from X into $C(X)$. Assuming this, we define the operator U by

$$U_x f = (T_x f)(T_x(g^x)) - T_x(fg^x), \quad f \text{ in } C(X),\ x \text{ in } X.$$

If $x_\alpha \to x$, then $g^{x_\alpha} \to g^x$ implies that $Uf(x_\alpha) \to Uf(x)$, so that Uf is continuous. It is clear that $U1 = 0$. Furthermore, $0 \leqq g^x \leqq 1$ implies that $T \pm U \geqq 0$. Finally

$$\begin{aligned} U_{sx} f &= T_{sx} f T_{sx}(g^{sx}) - T_{sx}(fg^{sx}) \\ &= T_x f_s T_x(g_s^{sx}) - T_x(f_s g_s^{sx}) \\ &= T_x f_s T_x(g^x) - T_x(f_s g^x) \\ &= U_x f_s \end{aligned}$$

since $g_s^{sx} = g^x$. Thus $T \pm U$ are in D; since $Ug(y) \neq 0$, we see that T is not extreme in D.

To show that g^x is continuous, let $z_\alpha \to z$. Then

$$\begin{aligned} |g^x(z_\alpha) - g^x(z)| &= \left| \int [\varphi(sx)(sz_\alpha) - \varphi(sx)(sz)]\, ds \right| \\ &\leqq \sup \{ |\varphi(sx)(sz_\alpha) - \varphi(sx)(sz)| : s \text{ is in } S\}. \end{aligned}$$

Consider the functions h_α defined on S by $h_\alpha(s) = |\varphi(sx)(sz_\alpha) - \varphi(sx)(sz)|$. Since the map S into $\varphi(sx)$ is continuous, h_α is continuous and it suffices to show that h_α converges to 0 uniformly on S. Suppose not; by taking subnets if necessary, we may suppose that $s_\alpha \to s$ and that there exists an $\varepsilon > 0$ such that $h_\alpha(s_\alpha) > \varepsilon$. But

$$\begin{aligned} h_\alpha(s_\alpha) &= |\varphi(s_\alpha x)(s_\alpha z_\alpha) - \varphi(s_\alpha x)(s_\alpha z)| \\ &\leqq |\varphi(s_\alpha x)(s_\alpha z_\alpha) - \varphi(sx)(s_\alpha z_\alpha)| \\ &\quad + |\varphi(sx)(s_\alpha z_\alpha) - \varphi(sx)(s_\alpha z)| \\ &\quad + |\varphi(sx)(s_\alpha z) - \varphi(s_\alpha x)(s_\alpha z)|. \end{aligned}$$

Since φ is continuous, the first and third terms tend to 0. Theorem A implies that the second term tends to 0. Hence g^x is continuous. Similarly the map $x \to g^x$ is continuous.

NOTATION. Let supp T denote the smallest closed set containing supp T_x for all x in X.

COROLLARY. *Let T be an element of D. If S is a group of homeomorphisms which is compact on* supp T, *then* (C) *is true.*

Proof. The proof proceeds as in Theorem 1 with a few technical changes due to the fact that supp T is not necessarily all of X.

NOTATION. If S is a group of homeomorphisms of X, let W be the set of all continuous maps from X into X which are pointwise limits of functions in S. If W is again a group, then we define W^x and $P(W^x)$ as we did for S. Theorem 2 below is a slight variant of Theorem 1. The hypothesis that S be a compact group is replaced by the weaker one that W be a compact group, while the conclusion is also weakened slightly since $P(W^x)$ is smaller than $P(S^x)$. In §5, we give an example which shows that the conclusion must necessarily be changed.

THEOREM 2. *Let S and W be as above. If W is a group and is compact on X, then T is extreme in D (if and) only if T_x is extreme in $P(W^x)$ for all x in X.*

Proof. Let $D(W)=\{T \text{ in } D_1 : T(f_w)=(Tf)_w \text{ for all } w \text{ in } W \text{ and } f \text{ in } C(X)\}$. From Theorem 1, we know that T is extreme in $D(W)$ if and only if T_x is extreme in $P(W^x)$ for all x in X. Hence we need only show that $D=D(W)$. Clearly $D(W)\subset D$.

Let $T\in D$, $f\in C(X)$ and $w\in W$. Let s_α in S be such that $s_\alpha \to w$ pointwise. By the corollary to Theorem A, $f_{s_\alpha}\to f_w$. Thus $T(f_{s_\alpha})$ converges to $T(f_w)$. But $(Tf)_{s_\alpha}$ converges to $(Tf)_w$ pointwise and $T(f_{s_\alpha})=(Tf)_{s_\alpha}$. Therefore $T(f_w)=(Tf)_w$. Since f and w were arbitrary, T is in $D(W)$ and the proof is complete.

A group S of homeomorphisms is said to be *transitive* on X if for every x and y in X there exists an s in S such that $sx=y$. In Theorem 3 below, we replace the assumption in Theorem 1 that S is a group by the weaker one that S is a semigroup of operators given by continuous functions from X into X which contains a subgroup S'. We replace the assumption that S be compact by the weaker one that S' be compact. But we have to add the restriction that S' is transitive on X. This last condition is very similar to the condition of A. Ionescu Tulcea that X be a group and $S=X$; the proof is essentially hers, with S' playing the role in our proof that left translation by members of X does in her proof.

THEOREM 3. *Let S be a semigroup of operators, each of which is given by a continuous function of X into X. Suppose that S contains a subset S' which is a group of homeomorphisms, compact on X. If S' is transitive, then conclusion* (C) *is true.*

Proof. Let T be an element of D and let y be an element of X such that T_y is not extreme in $P(S^y)$. We need to show that T is not extreme in D. Let μ be a measure such that $T_y \pm \mu$ are in $P(S^y)$. Define the operator U by

$$Uf(x) = \mu(f_s) \quad \text{where } sy = x \text{ and } s \text{ is in } S_1.$$

First we notice that U is well defined because if $sy=ty$, then $s^{-1}ty=y$ and thus $\mu(f_t)=\mu(f_s)$. Let s be any element of S and x any element of X. Then $U(f_s)(x)=\mu(f_{st})$ where $ty=x$, and $Uf(sx)=Uf(rx)=\mu(f_{rt})$ where $rx=sx$ and r is in S'. But $t^{-1}r^{-1}sty=y$ and thus $\mu(f_{rt})=\mu((f_{rt})_{t^{-1}r^{-1}st})=\mu(f_{st})$. Thus $U(f_s)=(Uf)_s$ for all s in S.

Next we show that Uf is continuous for all f in $C(X)$. Let x_α tend to x. Since S' is compact and transitive on X, we may assume that $x_\alpha = s_\alpha y$ and $x=sy$, where s_α tends to s. The corollary to Theorem A then implies that $Uf(x_\alpha)$ tends to $Uf(x)$. Since $\mu 1=0$, $U1=0$. Because $T_y \pm \mu \geqq 0$, $T \pm U \geqq 0$. Thus $T \pm U$ are in D. But $U \neq 0$, so T is not extreme in D.

5. **Examples.** By exhibiting a compact space X, a group S of homeomorphisms such that W is a compact group, an extreme operator T in D, and an x in X such that T_x is not extreme in $P(S^x)$, we show in Example 1 below that the conclusion of Theorem 2 is weaker and must be different from the conclusion of Theorem 1.

EXAMPLE 1. Let X be the circle group, i.e. $X=\{t : t \in R \pmod{2\pi}\}$. Let S be the group generated by σ and ρ where

$$\sigma(t) = t+1 \quad \text{and} \quad \rho(t) = -t, \qquad t \in X.$$

Let T be the operator defined by

$$Tf = (f_\sigma + f_{\sigma^{-1}})/2, \qquad f \text{ in } C(X).$$

It is easy to see that T is in D. If $x \neq n \pmod{\pi}$ for all integers, n, then $S^x=\{\text{identity}\}$. In this case, $P(S^x)=P_1$, the set of all probability measures on X. Since T_x is not a point mass, it is not extreme in $P(S^x)$ for such x. If $\sigma^z(t)=t+z$, then W, the closure of S, is $\{\sigma^x, \sigma^x\rho : x \in X\}$, since $\{n \pmod{2\pi}\}$ is dense in X. Thus, W^x is easily seen to be $\{\sigma^0, \sigma^{2x}\rho\}$. Since supp $T_x=\{\sigma x, \sigma^{-1}(x)\}=W^x(\sigma(x))$, T_x is extreme in $P(W^x)$ by Proposition 1. Thus T is extreme in D by Theorem 2 since W is compact on X.

Next we exhibit a compact metric space X, a group S of homeomorphisms of X isomorphic to the integers, and an operator T which is extreme in D such that T_x is not extreme in $P(S^x)$ for any x in X.

EXAMPLE 2. Let $X=\{(t, \lambda) : t \in R \pmod{2\pi}$ and $\lambda \in [0, 1]\}$ with the usual topology. Let S be the infinite cyclic group generated by s where

$$s(t, \lambda) = (t+\lambda\pi, \lambda), \qquad (t, \lambda) \in X.$$

Then $s^n(t, \lambda)=(t+n\lambda\pi, \lambda)$. Next we classify S^x. Let $x=(t, \lambda)$; then

$S^x=\{s^0\}$ if λ is irrational,

$S^x=\{s^{2nq} : n \text{ is an integer}\}$ if $\lambda=p/q$ in lowest terms and p is odd, and

$S^x=\{s^{nq} : n \text{ is an integer}\}$ if $\lambda=p/q$ in lowest terms and p is even.

Indeed, if $s^n(t, \lambda)=(t, \lambda)$, then $n\lambda=0 \pmod 2$, which implies that λ is rational, say $\lambda=p/q$, and q divides n while 2 divides np.

Now define T by

$$Tf(t, \lambda) = \frac{1}{2\pi}\int_0^{\pi t} f(\theta, \lambda^2)\, d\theta, \quad \text{for } f \text{ in } C(X),\ (t, \lambda) \text{ in } X.$$

Since $T1=1$, $T\geqq 0$ and $T(f_s)=Tf=(Tf)_s$, T is in D. Since S^x is clearly compact—indeed finite—on supp T_x and supp T_x is infinite, T_x is not extreme in $P(S^x)$ for any x in X by "(i) implies (iv)" of Proposition 1.

Suppose that T is not extreme in D. This implies that there exists an operator U, an x in X, and a g in $C(X)$ such that $0\leqq g\leqq 1$, $Ug(x)>0$ and $T\pm U$ are in D. Let $(\theta, \lambda)=x$. It is then possible to choose p/q in lowest terms such that p is odd, q is sufficiently large and $y=(\theta, p/q)$ is sufficiently close to x in order that

(i) $|Ug(y)-Ug(x)| < \frac{1}{4}Ug(x)$,

(ii) $|Tg(y)-Tg(x)| < \frac{1}{4}Ug(x)$,

(iii) $|1/q \sum_{n=1}^{n=q} g(\theta+\pi 2np^2/q, p^2/q^2)-Tg(x)| < \frac{1}{4}Ug(x)$, and

(iv) $|t_1-t_2| < 2/q \pmod{2\pi}$ implies $|g(t_1, p^2/q^2)-g(t_2, p^2/q^2)| < \frac{1}{4}Ug(x)$.

The first two are possible since Ug and Tg are continuous. The last two are possible since g is uniformly continuous and the number $Tg(x)$ is the limit of sums of the form appearing in (iii).

We intend to use inequalities (i)–(iv) to obtain a measure μ, a function h and a number b such that

$$(\mu-1/3U_y)(h+b) > 0 \quad \text{and} \quad (\mu-1/3U_y)(h+b) < 0.$$

This will contradict the assumption that U exists and prove that T is extreme in D.

Define μ by

$$\mu f = 1/q \sum_{n=1}^{n=q} f(\theta+2np^2\pi/q, p^2/q^2), \qquad f \text{ in } C(X).$$

Define h by

$$h(t, \lambda') = 1/q \sum_{n=1}^{n=q} g(t+2np^2\pi/q, p^2/q^2) \quad \text{for all } (t, \lambda') \text{ in } X.$$

Notice that $\mu h=\mu g=h(y)$. Now $T\pm U\geqq 0$ implies that supp U_y is contained in supp T_y. But, on supp T_y, $h=1/q \sum_{n=1}^{n=q} g_s 2nq$ and s^{2nq} is in S^y. Thus $Ug(y)=Uh(y)$ and $Tg(y)=Th(y)$.

From inequality (iv) we derive the fact that $\max\{h(z): z\in X\}-\min\{h(z): z\in X\} < \frac{1}{4}Ug(x)$. Let $b=\frac{1}{4}Ug(x)-h(y)$. Then $h+b\geqq 0$ so $(T_y-U_y)(h+b)\geqq 0$. Since

$$\begin{aligned}(\mu-1/3U_y)(h+b) &= \mu h-1/3Uh(y)+b\\ &= \mu g-1/3Ug(y)+b\\ &> Tg(x)-\tfrac{1}{4}Ug(x)-1/3Ug(y)+b \quad \text{from (iii)}\\ &> Tg(y)-\tfrac{1}{2}Ug(x)-1/3Ug(y)+b \quad \text{from (ii)}\\ &> Tg(y)-\tfrac{1}{2}(4/3Ug(y))-1/3Ug(y)+b \quad \text{from (i)}\\ &= (T_y-U_y)(h+b) \geqq 0,\end{aligned}$$

we have

$$(\mu - 1/3U_y)(h + b) > 0.$$

But

$$\begin{aligned}(\mu - 1/3U_y)(h+b) &= \mu h + b - 1/3Uh(y) \\ &= h(y) + \tfrac{1}{4}Ug(x) - h(y) - 1/3Uh(y) \\ &< \tfrac{1}{4}Ug(x) - 1/3(3/4Ug(x)) = 0 \quad \text{from (i).}\end{aligned}$$

Hence $0 < (\mu - 1/3U_y)(h+b) < 0$, which was to be shown.

REMARK. Example 2 also shows that the condition that S be compact cannot be replaced by the condition that S be closed, i.e. that $S = W$. This is because $S = W$ in Example 2. To show this, we suppose s^{n_α} converges to w pointwise on X, where W is a continuous function from X into X. Since $s^{n_\alpha}(\theta, \lambda) = (\theta + \lambda n_\alpha \pi, \lambda)$, it follows that there is a real number r such that $w(\theta, \lambda) = (\theta + \lambda r\pi, \lambda), (\theta, \lambda) \in X$. Then $w(0, 1) = (r\pi, 1) = \lim (n_\alpha \pi, 1) = (0, 1)$ or $(\pi, 1)$ implies that $r = 0$ or 1 (mod 2). Thus r is an integer, and $w \in S$. Since w was an arbitrary element of W, we have $W = S$.

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